A small tour of optimization models

Theory of games and markets with examples

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Walras problem

Denote $y = (y_i,\ i = 1, ..., m),\ p = (p_i,\ i = 1, ..., m),\ x = (x_{ij},\ i, j = 1, ..., m)$. Here x_{ii} are local resources, x_{ij} are server j resources used by i. Profit of i

$$u_i(y, p, x) = a_i y_i + p_i \sum_{j \neq i} x_{ji} - \sum_{j \neq i} p_j x_{ij}, \tag{1}$$

Customer expences

$$h_i = y_i + \gamma_i,\ \gamma_i = n_i / w_i,\ w_i = c_{0i}(1 - e^{-c_{ii}x_{ii} - c_{ij}x_{ij}}), \tag{2}$$

where w_i is capacity of server i, b_i is resource of server i, c_{ij} defines efficiency of resources, here local resources x_{ii} are defined by balance condition:

$$x_{ii} + \sum_j x_{ij} = b_i,\ i = 1, ..., m, \tag{3}$$

Optimal contract in resources x, m=2

Denote the contract resources as $\bar{x}_{12}$ and $\bar{x}_{21}$. Then "no-contract" resources:

$$\bar{\bar{x}}_{12}(y,p) = \arg\max_{x_{12}} u_1(y,p,x_{12},\bar{x}_{21}) \tag{4}$$

$$\bar{\bar{x}}_{21}(y,p) = \arg\max_{x_{21}} u_2(y,p,x_{21},\bar{x}_{12}) \tag{5}$$

Denote

$$\begin{aligned} f_x(y,p,\bar{x}_{12},\bar{x}_{21}) = {} & u_1(y,p,\bar{\bar{x}}_{12},\bar{x}_{21}) - u_1(y,p,\bar{x}_{12},\bar{x}_{21}) \\ & + u_2(y,p,\bar{\bar{x}}_{21},\bar{x}_{12}) - u_2(y,p,\bar{x}_{21},\bar{x}_{12}) \end{aligned} \tag{6}$$

Contract $(\bar{x}_{12}(y,p),\bar{x}_{21}(y,p))$ is stable if

$$\min_{\bar{x}_{12},\bar{x}_{21}} f_x(y,p,\bar{x}_{12},\bar{x}_{21}) = 0, \tag{7}$$

Optimal resources: $x^\star_{12} = x_{12}(y,p)$ and $x^\star_{21} = x_{21}(y,p)$.

Optimal contract in prices (y,p), m=2

Denote the contract prices $(\bar{y}_1, \bar{p}_1, \bar{y}_2, \bar{p}_2)$. Then the "no-contract" prices:

$$(\bar{\bar{y}}_1, \bar{\bar{p}}_1) = \arg\max_{(y_1, p_1)} u_1(y_1, \bar{y}_2, p_1, \bar{p}_2, x_{12}(\bar{y}_2, \bar{p}_2), x_{21}(y_1, p_1)), \quad (8)$$

$$(\bar{\bar{y}}_2, \bar{\bar{p}}_2) = \arg\max_{(y_2, p_2)} u_2(y_2, \bar{y}_1, p_2, \bar{p}_1, x_{12}(y_2, p_2), x_{21}(\bar{y}_1, \bar{p}_1)) \quad (9)$$

$$\min_{\bar{y}_1, \bar{p}_1, \bar{y}_2, \bar{p}_2} f(\bar{y}_1, \bar{p}_1, \bar{y}_2, \bar{p}_2) = 0, \quad (10)$$

$$\begin{aligned} f(\bar{y}_1, \bar{p}_1, \bar{y}_2, \bar{p}_2) = & \quad (11) \\ u_1(\bar{\bar{y}}_1, \bar{y}_2, \bar{\bar{p}}_1, \bar{p}_2, x_{12}(\bar{y}_2, \bar{p}_2), x_{21}(\bar{\bar{y}}_1, \bar{\bar{p}}_1)) - \\ u_1(\bar{y}_1, \bar{y}_2, \bar{p}_1, \bar{p}_2, x_{12}(\bar{y}_2, \bar{p}_2), x_{21}(\bar{y}_1, \bar{p}_1)) + \\ u_2(\bar{\bar{y}}_2, \bar{y}_1, \bar{\bar{p}}_2, \bar{p}_1, x_{12}(\bar{\bar{y}}_2, \bar{\bar{p}}_2), x_{21}(\bar{y}_1, \bar{p}_1) - \\ u_2(\bar{y}_2, \bar{y}_1, \bar{p}_2, \bar{p}_1, x_{12}(\bar{y}_2, \bar{p}_2), x_{21}(\bar{y}_1, \bar{p}_1). \end{aligned}$$

Walras problem, graph error

Relation of profit $u_1(p_1)$ to resource price p_1
(other variables are fixed as the contract prices $(\bar{y_1}, \bar{y_2}, \bar{p_2})$):

Correct relation:

$$u_1(p_1) = a_1\bar{y_1} + p_1 x_{21}(\bar{y_1}, \bar{y_2}, p_1, \bar{p_2}) - p_2 x_{12}(\bar{y_1}, \bar{y_2}, p_1, \bar{p_2}), \quad (12)$$

Observed error:

$$u_1(p_1) = a_1\bar{y_1} + p_1 x_{21}(\bar{y_1}, \bar{y_2}, \bar{p_1}, \bar{p_2}) - p_2 x_{12}(\bar{y_1}, \bar{y_2}, \bar{p_1}, \bar{p_2}), \quad (13)$$

(the same error is in $u_2(p_2)$).

Walras problem, optimization error

Denote the contract prices $(\bar{y_1}, \bar{p_1})$.

Correct "no-contract" prices:

$$(\bar{\bar{y_1}}, \bar{\bar{p_1}}) = \arg\max_{(y_1, p_1)} (a_1 y_1 + p_1 x_{21}(y_1, \bar{y_2}, p_1, \bar{p_2}) - \bar{p_2} x_{12}(y_1, \bar{y_2}, p_1, \bar{p_2})), \quad (14)$$

Suspected error:

$$(\bar{\bar{y_1}}, \bar{\bar{p_1}}) = \arg\max_{(y_1, p_1)} (a_1 y_1 + p_1 x_{21}(y_1, \bar{y_2}, \bar{p_1}, \bar{p_2}) - \bar{p_2} x_{12}(y_1, \bar{y_2}, \bar{p_1}, \bar{p_2})), \quad (15)$$

(the same error is suspected in $(\bar{y_2}, \bar{p_2})$).